

ON THE SEMI-THREADING OF KNOT DIAGRAMS WITH MINIMAL OVERPASSES

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ABSTRACT. Given a knot diagram D , we construct a semi-threading circle for it which can be an axis of D as a closed braid depending on knot diagrams. In particular, we consider semi-threading circles for minimal diagrams of a knot with respect to overpasses which give us some information related to the braid index. By this notion, we show that, for every nontrivial knot K , the braid index $b(K)$ of K is not less than the minimum number $l(K)$ of overpasses of diagrams. Moreover, they are the same for a torus knot.

1. INTRODUCTION

Throughout this paper, every knot is oriented and lies in the 3-dimensional sphere S^3 . Also, all knots are isotopic to polygonal or smooth knots, i.e., they are *tame*. Therefore, each knot has a diagram with finitely many crossings, hence, has a finite number of overpasses. A diagram D of a knot K is a popular representative of the isotopy class of K which is also called the *knot type* of K . D is obtained from a regular projection P of K by the following steps. First, we take a sufficiently small neighborhood of each double point of P so that the intersection of the neighborhood and P looks like an ‘X-shape’ on the plane. Second, we adjust the interior of each neighborhood so that we have a knot D which is isotopic to K and regularly projected to P . In this sense, a knot diagram is an ‘almost planar’ knot, i.e., it lies in the plane except for a small neighborhood of each double point of the regular projection. It has been found various diagrams of knot types whose numbers of crossings are minimal. Here, we prefer the number of overpasses of a knot diagram and are interested in knot diagrams whose numbers of overpasses are minimal. This gives us another point of view to consider knot diagrams. Notice that the minimal number of overpasses of a knot is the classical *bridge number* and first studied by Schubert in [7], where the effect of various operations on knots (satellite, cabling, connected sum) on this number was investigated. We refer to [1] for further studies. The first author investigated the relationships between the number of crossings and the number of overpasses of a knot diagram. It turns out the number of overpasses is estimated by that of crossings if the knot diagram has a minimal number of crossings [3]. On the other hand, we

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define a ‘semi-threading’ of an oriented knot diagram and mainly consider ‘minimal semi-threading’ based on our definition. From this construction, we know that the braid index of a nontrivial knot is not less than the minimum number of overpasses of its diagrams. This theorem suggests important information about minimal knot with respect to overpasses. In particular, the braid index of each torus knot is the same as the minimum number of overpasses of its diagrams. That is, the semi-threading circle is just a threading. For a ‘threading’ of a knot in detail, see [2].

2. MINIMAL CROSSINGS AND MINIMAL OVERPASSES OF KNOT TYPE

Given a knot diagram D , the number of crossings or the crossing number of D is denoted by $c(D)$. For each knot K , we denote $\min\{c(D) \mid D \text{ is a diagram of } K\}$ by $c(K)$. Note that we may assume a knot diagram D lies in the plane by indicating ‘overcrossings’ and ‘undercrossings’. A ‘crossing’, in this sense, of a knot diagram D means a ‘signed double point’ of the regular projection of D . Hence, $c(D)$ is the number of all double points of the regular projection of D . On the other hand, we may regard a crossing of D as the pair of two points overcrossing and undercrossing in D which are projected to the same double point. That is, a crossing is considered as the pre-image of a double point under the projection map.

Proposition 2.1. *Let D be a knot diagram. Then there are a unique nonnegative integer k and a finite sequence $s_1, f_1, s_2, f_2, \dots, s_k, f_k$ of $2k$ points of D which are neither overcrossings nor undercrossings of D such that*

$$[s_1, f_1], [s_2, f_2], \dots, [s_{k-1}, f_{k-1}], [s_k, f_k]$$

and

$$[f_1, s_2], [f_2, s_3], \dots, [f_{k-1}, s_k], [f_k, s_1]$$

are the overpasses and the underpasses of D with respect to the sequence, respectively, where $[s_i, f_i]$, for each $i \in \{1, \dots, k\}$, is the closed arc of D from s_i to f_i which contains at least one overcrossing but has no undercrossing; similarly, $[f_i, s_{i+1}]$, for each $i \in \{1, \dots, k-1\}$, is the closed arc of D from f_i to s_{i+1} which contains at least one undercrossing but has no overcrossing; also, the last one $[f_k, s_1]$ is the closed arc of D from f_k to s_1 which contains at least one undercrossing but has no overcrossing.

In Proposition 2.1, such a sequence $s_1, f_1, s_2, f_2, \dots, s_k, f_k$ is called an *over-underpass sequence* of D . Since any over-underpass sequence of D consists of $2k$ points, the number of overpasses (or underpasses) with respect to any over-underpass sequence of D is k . Hence, we can define the number of overpasses (or underpasses) of the knot diagram D as k , and denote it by $l(D)$, called the length of over-underpass sequences or the bridge number of D . Also, for each knot K , we denote $\min\{l(D) \mid D \text{ is a diagram of } K\}$ by $l(K)$.

Notice that (1) $c(D)$ and $l(D)$ are plane isotopy invariants of knot diagrams, i.e., if D_1 and D_2 are plane isotopic knot diagrams, then $c(D_1) = c(D_2)$ and $l(D_1) = l(D_2)$;

(2) $c(K)$ and $l(K)$ are isotopy invariants of knots, i.e., if K_1 and K_2 are isotopic knots, then $c(K_1) = c(K_2)$ and $l(K_1) = l(K_2)$.

Corollary 2.2. *If $l(D) \leq 1$, then D is a diagram of a trivial knot. Therefore, a knot K is trivial if and only if K has a diagram D with $l(D) \leq 1$.*

Obviously, the converse is not true. A diagram of a double twisted circle can be an example. Similarly, for any positive integer k , there is a diagram of a trivial knot whose number of overpasses is greater than k . On the other hand, given a knot diagram D with at least one crossing, we can add crossings to D as many as we want without changing the knot type and the number of overpasses of D . Take a sufficiently small arc of D from s_1 to a point between s_1 and the first overcrossing of D from s_1 and twist it alternately so that the number of overpasses of D is not changed. Or, we may modify the interior of a sufficiently small neighborhood of a crossing of D . Hence, we have the following corollary.

Corollary 2.3. *If D is a diagram of a knot K such that $c(D) \geq 1$, then for every positive integer n , there is a diagram D' of K such that $l(D') = l(D)$ and $c(D') \geq c(D) + n$.*

Lemma 2.4. *$l(D) \leq c(D)$ for any knot diagram D . The equality holds if D is an alternating knot diagram. Furthermore, $l(K) \leq c(K)$ for any knot K .*

Remark that, since the number of double points of D is finite and we have an ϵ -neighborhood of D for a sufficiently small ϵ which is the regular projection of a knotted solid torus whose axis is the knot diagram D , we can always take such an ϵ and such ϵ -neighborhoods as above.

The following theorem gives us a relationship between the numbers of crossings and overpasses.

Theorem 2.5. [3]. *If D is a minimal diagram of a nontrivial knot K with respect to crossings, then $1 + \sqrt{1 + c(D)} \leq l(D) \leq c(D)$.*

Notice that a knot K need not hold $1 + \sqrt{1 + c(D)} \leq l(D)$. The closure of the braid b_1^5 , where b_1 is the generator of the standard group presentation of the braid group B_2 , can be shown as an example for it by the Theorem 3.6 in the next section. Also, remark that the crossing number of a knot diagram with a minimal number of overpasses can be arbitrarily large as shown at Corollary 2.3.

3. SEMI-THREADING KNOT DIAGRAM AND BRAID INDEX

In this section, we briefly introduce a definition of threading of a knot diagram. For details, see [2]. Also, we define a ‘semi-threading’ of a knot diagram. In particular, we consider the semi-threading of minimal diagrams of a knot with respect to overpasses.

First, let us introduce Alexander braiding theorem and Markov theorem for braids shortly.

(1) Alexander Braiding Theorem: Every link is the closure of a braid, that is, a closed braid.

This theorem is published by J.W. Alexander in 1923 and allows one to study knots and links using the theory of braids [4]. The minimum number of braid strands used in the closure is called the *braid index* of the link. Note that the closures of different braids need not be distinct. An answer for this question is the following theorem.

(2) Markov Theorem: The closures of two braids are isotopic if and only if one braid can be obtained from the other by a finite sequence of *Markov moves*, which are sometimes called conjugations and stabilizations, respectively [5].

Definition 3.1. [2]. Let K be a knot, T a trivial knot, and $K \sqcup T$ an oriented link in S^3 whose components are only K and T and the linking number $lk(K, T)$ is positive. The link $K \sqcup T$ is said to be *braided* if there are isotopic diagrams K' , T' , $K' \sqcup T'$ of K , T , $K \sqcup T$, respectively, such that K' is a closed braid, i.e, a closure of a braid, and T' is an axis of K' , which is called a *threading circle* of K' . The link diagram $K' \sqcup T'$ is called a *threading* of K' .

By the definition of a threading circle of a knot diagram, we easily know that, for each knot K , the braid index $b(K)$ is the same as $\min\{lk(D, L) \mid D \text{ is a diagram of } K \text{ and } L \text{ is a threading circle of } D\}$. Notice that $b(K)$ is an isotopy invariant of knots.

Actually, a threading circle of a knot diagram is an axis of the knot as a closed braid. We define a ‘weaker threading’ as follows.

Definition 3.2. Let D be a diagram of a knot. An oriented simple closed curve L on the plane is called a *semi-threading circle* of D if L crosses all overpasses of D transversely exactly once so that the linking number $lk(D, L)$ of D and L is positive. We call such a link $D \sqcup L$ a *semi-threading* of D .

Now we construct a semi-threading circle for a knot diagram which might be an axis of D as a closed braid depending on knot diagrams. However, a threading circle of a knot diagram need not be a semi-threading circle of it because a semi-threading circle requires all the overpasses.

Theorem 3.3. *A semi-threading circle L of a knot diagram D exists.*

Proof. If D is a diagram of a trivial knot with no crossing, we can draw an oriented simple closed curve L on the plane so that $D \sqcup L$ is a diagram of a positive Hopf link, a two component link whose components are trivial and linking number is 1. Obviously, L is a semi-threading circle of D .

Suppose $l(D) = k \geq 2$ and $s_1, f_1, s_2, f_2, \dots, s_k, f_k$ is an over-underpass sequence of D . By a plane isotopy, we can arrange all overpasses $[s_1, f_1], [s_2, f_2], \dots, [s_k, f_k]$

to be vertical downward and from left to right, i.e., we have vertical overpasses $[s_1, f_1], [s_2, f_2], \dots, [s_k, f_k]$ from left to right.

Now let us construct a simple closed curve L on the plane to be a semi-threading circle of D . For each $i \in \{1, \dots, k\}$, choose a point x_i on (s_i, f_i) which is not a crossing of D , where $(s_i, f_i) = [s_i, f_i] - \{s_i, f_i\}$. Take a sufficiently small circular closed neighborhood of x_i in the plane so that the intersection of the neighborhood and D is a closed arc contained in (s_i, f_i) which has no crossing of D . Choose two points x'_i and x''_i on the boundary of the neighborhood so that the line segment $[x'_i, x''_i]$ is perpendicular to $[s_i, f_i]$ and passes through x_i from left to right. Draw the straight line l_i from x''_i to x'_{i+1} for each $i \in \{1, \dots, k-1\}$. If l_i tangents D , by a sufficiently small change, we can modify l_i to a piecewise straight line which intersects D transversely and does not intersect the overpasses. We also denote the piecewise straight line by l_i . By the above construction, we have a piecewise straight line

$$L_0 = [x'_1, x''_1] \cup l_1 \cup \dots \cup [x'_{k-1}, x''_{k-1}] \cup l_{k-1} \cup [x'_k, x''_k]$$

from x'_1 to x''_k which intersects D transversely and intersects each overpass exactly once. Let ϵ be a sufficiently small positive real number such that

- (1) $L_0 + \epsilon$ is the parallel transition of L_0 by ϵ along the overpasses;
- (2) $L_0 + \epsilon$ does not pass through any crossings of D ;
- (3) $L_0 + \epsilon$ intersects D transversely;
- (4) there is no crossing of D between L_0 and $L_0 + \epsilon$;
- (5) each of the vertical line segments $[x'_1, x'_1 + \epsilon]$ and $[x''_k, x''_k + \epsilon]$ does not intersect D . Now we get a piecewise straight line l'_k from x''_k to x'_1 as

$$l'_k = [x'_1, x'_1 + \epsilon] \cup (L_0 + \epsilon) \cup [x''_k, x''_k + \epsilon]$$

and a simple closed curve $L_0 \cup l'_k$ on the plane. Let $L = L_0 \cup l'_k$ and give the orientation which agrees with from x'_1 to x''_k clockwise.

As the next step, we give the crossing structures to the intersections of D and L to make the linking number $lk(D, L)$ positive. For each $i \in \{1, \dots, k\}$, $[x'_i, x''_i]$ crosses $[s_i, f_i]$ below, and, for each $i \in \{1, \dots, k-1\}$, l_i crosses above the underpasses of D which intersect l_i . In particular, l'_k crosses above at any intersection with D . Therefore, L is a semi-threading circle of D and the link diagram $D \sqcup L$ is a semi-threading of D . \square

In the proof of Theorem 3.3, such a trivial knot diagram L is unique up to isotopy and called a *minimal semi-threading circle* of D . Also, we denote it by M_D . The semi-threading $D \sqcup M_D$ of D is called a *minimal semi-threading* of D . Remark that we can construct a semi-threading circle without changing the knot diagram. In other words, the plane isotopy to make all overpasses vertical downward is not necessary. It is obvious from the construction of the semi-threading circle in the above proof.

We have already had well-known threadings devised by Morton [2]. They are also very good examples of threadings. As an advantage of minimal semi-threading, it immediately shows the relationship between the number of overpasses and the linking number of threading, Corollary 3.4. On the other hand, Morton's threading depends on unknotted simple closed curves and numbers of crossings between knot and unknot diagrams, but in his paper [2], Morton proved valuable theorems on his own threading.

In general, a threading circle of a knot diagram need not require all overpasses of it. However, each minimal semi-threading of $D \sqcup M_D$ and the threading $D \sqcup \text{Morton's}$ for a knot diagram D still require all overpasses of D . For our purpose on this paper, the notion of minimal semi-threading is necessary as a key fact. The following statement is an immediate consequence of the minimal semi-threading of a knot diagram.

Corollary 3.4. *If D is a knot diagram with $l(D) \geq 1$, then $l(D) = lk(D, M_D)$.*

The following lemma plays an important role between the minimal number of overpasses and the braid index of a knot type. It may be regarded as another version of Alexander Braiding Theorem.

Lemma 3.5. *If D is a knot diagram and a minimal semi-threading $D \sqcup M_D$ is not braided, then there is a threading circle T of D such that $lk(D, M_D) < lk(D, T)$.*

Proof. Suppose that D is a knot diagram and $l(D) = k \geq 2$ and $s_1, f_1, s_2, f_2, \dots, s_k, f_k$ is an over-underpass sequence of D . By Corollary 3.4, $lk(D, M_D) = k$. Let us use L_0 in the proof of Theorem 3.3. That is,

$$L_0 = [x'_1, x''_1] \cup l_1 \cup \dots \cup [x'_{k-1}, x''_{k-1}] \cup l_{k-1} \cup [x'_k, x''_k].$$

We draw rays l_{L_0} and r_{L_0} starting from x'_1 and x''_k leftward and rightward which extend the line segments $\overline{x'_1, x''_1}$ and $\overline{x'_k, x''_k}$, respectively, and cross above the underpasses of D which intersect them. Let $T_0 = l_{L_0} \cup L_0 \cup r_{L_0}$. We can think of T_0 as a simple closed curve passing through the infinity ∞ of S^3 . Hence, $D \sqcup T_0$ is isotopic to $D \sqcup M_D$. We call T_0 an extended minimal semi-threading circle of D .

Our goal is to modify the extended minimal semi-threading circle T_0 to a threading circle T so that $D \sqcup T$ is braided. Note that, for each $i \in \{1, \dots, k\}$, the i -th underpass $[f_i, s_{i+1}]$ or $[f_k, s_1]$ crosses below T_0 an odd times. Notice that the extended semi-threading $D \sqcup T_0$ is really a threading, that is, $D \sqcup T_0$ is braided if each underpass of D crosses below T_0 exactly one time.

By hypothesis, $D \sqcup T_0$ is not braided, so D has at least one underpass which crosses T_0 below more than one time.

Suppose that $1 \leq m \leq k$ and $\{u_1, \dots, u_m\}$ is the set of all underpasses of D each of which crosses T_0 below more than one time. For each $i \in \{1, \dots, m\}$, there is $n_i \in \mathbb{N}$ such that u_i crosses T_0 below exactly $2n_i + 1$ times. Give the order to $2n_i + 1$ undercrossings on u_i by T_0 which agree with the orientation of D . We do crossing

change for all the even numbered undercrossings. That is, n_i times of crossing change occur. Hence, exactly $n_1 + \cdots + n_m$ times of crossing change occur on T_0 . Let T be this modified T_0 . Then $D \sqcup T$ is braided with $lk(D, T) = lk(D, M_D) + n_1 + \cdots + n_m$. This proves the lemma. \square

Notice that, in the proof of Lemma 3.4, $D \sqcup M_D$ and $D \sqcup T$ are surely not isotopic. Even though the threading $D \sqcup T$ is a diagram of a braided link, it may have lots of unnecessary strands as a closed braid. These unnecessary strands can be reduced by a suitable sequence of Markov moves.

Now we show an inequality between the number of minimal overpasses and the braid index for a nontrivial knot. From now on, we consider only minimal knot diagrams with respect to overpasses.

Theorem 3.6. *If K is a nontrivial knot, then $l(K) \leq b(K)$.*

Proof. Let K be a nontrivial knot, and let D be a minimal diagram of K with respect to overpasses. Hence, $l(D) = l(K)$. Consider an extended minimal semi-threading $D \sqcup T_0$ of D . If $D \sqcup T_0$ is braided, then $b(K) = lk(D, T_0)$, so $l(K) \leq b(K)$. Suppose that $D \sqcup T_0$ is not braided. Using plane isotopy, we remove all unnecessary crossings between the underpasses of D and the extended minimal semi-threading circle T_0 keeping on the structure of over-underpass sequence of D . Assume that D' is the modified diagram of D . By Corollary 3.4 and Lemma 3.5, $l(K) = lk(D', T_0) \leq lk(D', T)$, where T is such a threading circle of D' modified from T_0 as in the proof of Lemma 3.5. Since all unnecessary crossings between the underpasses and T_0 are removed, the threading $D' \sqcup T$ has no unnecessary strands as closed braid. Therefore, $b(K) = lk(D', T)$. This proves the theorem. \square

In general, for a knot K , it is not true that $l(K) = b(K)$. As an example, the figure 8-knot K has $l(K) = 2$ but $b(K) = 3$. Our main concern is what conditions for a knot K make $l(K) = b(K)$. One of such special classes of knot types is the torus knots.

Theorem 3.7. *If K is a torus knot, then $l(K) = b(K)$.*

Proof. Suppose that K is a (p, q) -torus knot, where p and q are integers which are relatively prime. Let D be a standard diagram of K . Without loss of generality, we may assume that $0 < p < q$. Then $c(D) = pq - q$ and $l(D) = q$. In this case, $c(K) = c(D)$ by [6] and $b(K) = b(D) = p$. By Theorem 3.6, $l(K) \leq b(K)$. Since K is a (p, q) -torus knot, K is isotopic to a (q, p) -torus knot K' . Let D' be a standard diagram of K' . Then $c(D') = pq - p$ and $l(D') = p$. Since the minimal number of crossings or overpasses of knot diagrams is an isotopy invariant of knots, $c(K') = c(K) = c(D) < c(D')$ but $l(K) = l(K') = l(D') = p$. Therefore, $b(K) = l(K) = p$. \square

This theorem says that a knot diagram with a minimal number of overpasses gives a piece of information of the braid index of the knot type. In other words, given a diagram D of a knot K , try to remove all unnecessary overpasses of D . When we can do so, we will approach the braid index. In general, it is very hard to find $l(K)$ and $b(K)$ for a knot K . Hence, it is very valuable that we know more exact relationship between $l(K)$ and $b(K)$.

4. REPRESENTING THREADING KNOT DIAGRAM AS A CLOSED BRAID

In this section, we explain how to get an isotopic closed braid from our threading of a knot diagram.

Consider a threading $D \sqcup T$ of a knot diagram D with a threading circle T . Let $s_1, f_1, s_2, f_2, \dots, s_k, f_k$ be an over-underpass sequence of D . We may assume all overpasses of D lie in $R_+^3 = \{(x, y, z) \in R^3 \mid z > 0\}$ and all underpasses of D are on the plane, i.e., the xy -plane of R^3 .

By an isotopy, change the i -th overpass $[s_i, f_i]$ for each $i \in \{1, \dots, k\}$ to the semi-circle $\widehat{s_i, f_i}$ in R_+^3 from s_i to f_i whose projection is that of $[s_i, f_i]$ and T to a straight line T' on the plane which intersects perpendicularly to the projection of each semi-circle, i.e., we can think of T' as a simple closed curve passing through the infinity ∞ of S^3 .

Next, we modify the underpasses of D to get the desired knot. Now the plane contains only T' and the underpasses of D . Take a positive real number α and push down all underpasses of D by α so that they are on the plane $z = -\alpha$.

For each $i \in \{1, \dots, k-1\}$, let $u_i = [f_i, s_{i+1}] - \alpha$ and $u_k = [f_k, s_1] - \alpha$, where u_i is the parallel transition of the i -th underpass by $-\alpha$, and, for each $i \in \{1, \dots, k\}$, let l_{s_i} and l_{f_i} be the vertical line segments from $s_i - \alpha$ to s_i and from f_i to $f_i - \alpha$, respectively. Then

$$(u_1 \cup u_2 \cup \dots \cup u_k) \cup (l_{s_1} \cup l_{s_2} \cup \dots \cup l_{s_k})$$

represents a braid. To show this, fix the points $f_1 - \alpha, \dots, f_k - \alpha$ and lift up l_{s_1}, \dots, l_{s_k} . Then we can get the braid. Now let K' be

$$(\widehat{s_i, f_i} \cup \dots \cup \widehat{s_k, f_k}) \cup (l_{f_1} \cup \dots \cup l_{f_k}) \cup (u_1 \cup \dots \cup u_k) \cup (l_{s_1} \cup \dots \cup l_{s_k}).$$

Then D represents K' and $D \sqcup T$ also represents $K' \sqcup T'$ as desired.

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